The Correlation between Thermal Resistance and Characteristic Impedance of Microwave Transmission Lines

Power dissipated in the center conductor of a transmission line causes an increase in temperature, which is the determining factor in the power handling capability. Mathematical solutions for the thermal resistance of the center conductor relative to the outer conductor and the electrical solution of the characteristic impedance Z_0 are similar; however dielectric loss is not included and, therefore, makes the similarity invalid. A simple technique is described that includes dielectric loss and preserves similarity, thus achieving an overall correlation between thermal resistance and Z_0. The technique is applied to many important practical transmission line configurations.

The power handling capability of microwave transmission lines is based on the analysis of heat conduction in the cross-sectional region between the inner and outer conductors. The main parameter of interest is the temperature increase T_o of the inner conductor with respect to the outer conductor. Because of the linearity of the heat equation, T_o can be expressed as the superposition of two components:

$$T_o = T_{o(c)} + T_{o(d)}$$

where

$$T_{o(c)} = R_{o(c)}P_c$$
$$T_{o(d)} = R_{o(d)}P_d \qquad (1)$$

$T_{o(c)}$ and $T_{o(d)}$ are the parts of T_o caused by the thermal dissipation in the inner conductor, P_c, and in the dielectric, P_d. $R_{o(c)}$ is the thermal resistance associated with heat flow from the inner conductor through the dielectric region to the external conductor. $R_{o(d)}$ is the thermal resistance associated with heat flow from the dielectric region to the external conductor where the heat is generated in the dielectric region due to dielectric loss tangent.

The distinguishing features of a given microwave transmission line are the cross-sectional shape of the region between the center conductor and the outer conductor and the dielectric material used. The analysis of the electric and thermal fields in the cross-sectional region may be very complex, particularly if the center conductor has a different shape than the outer conductor. Such cases can be more of a scientific investigation into the solution of partial differential equations rather than an attempt at solving the basic problem at hand. There are publications[1]

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that list standard transmission line configurations along with solutions usually obtained by conformal mapping, but to the author's knowledge there is no apparent general listing in the literature of all types of configurations. The references on heat transfer[2] can supply solutions for additional configurations such as tube in tube, the tube in filled form and the strip in filled form.

Once having obtained a long list of standard solutions, expressions can be developed for Z_0 and $R_{o(c)}$. However, expressions for $R_{o(d)}$ still are not available. As mentioned previously, power dissipated within the dielectric will add to the temperature increase in the transmission line. Accordingly, $R_{o(d)}$ accounts for the temperature increase due to heat generated from within the dielectric and flowing to the external conductor. This component of the power handling capability of transmission lines has only been covered in the literature in one instance — a Russian-language journal.[3] This article uses the results of that reference to relate dielectric loss to Z_0 and applies the results to many transmission line configurations.

For the condition where the external conductor is maintained at constant temperature, it will be shown that the overall correlation between the thermal resistances $R_{o(c)}$ and $R_{o(d)}$ and impedance Z_0 does indeed exist. It is not obvious that $R_{o(d)}$ would be included in this result.

THEORY — BEFORE ADDING DIELECTRIC LOSS

It is known[1] that for the basic TEM mode in the cross-section S of a transmission line filled with dielectric, the electric field is expressed by Laplace's equation:

$$\Delta\phi = 0 \text{ with } \phi\,|_{ib} = \phi_c \text{ and } \phi\,|_{eb} = 0 \tag{2}$$

where

Δ = Laplace's operator
ϕ = instantaneous value of electric potential at any point in the cross-section S and a two-dimensional function over S

The conductors forming the inner and external boundaries of S are assumed to have high conductivity and, therefore, will coincide with the contours of equal potential such as $\phi = \phi_c$ and $\phi = 0$.

Equation 2 assumes charge is located on the inner and outer conductors, and not in the region S. The analogous heat equation is expressed as

$$\Delta T = 0 \text{ with } T\,|_{ib} = T_{o(c)} \text{ and } T\,|_{eb} = 0 \tag{3}$$

Since differential equations (Equations 2 and 3) have the same form, it follows that ϕ and T are similar,

$$\frac{T}{\phi} = \frac{T_{o(c)}}{\phi_c}$$

and congruent,

$$\oint_C \frac{(\text{grad}\,T, \vec{n})dl}{T_{o(c)}} = \oint_C \frac{(\text{grad}\,\phi, \vec{n})dl}{\phi_c} \tag{4}$$

where

C = any contour enclosing the inner conductor
$\vec{n}$ = unit vector, normal to the contour, outward from the enclosing surface
dl = elemental length of the contour

Another way to view Equation 4 is that the two types of solutions, ϕ and T, normalized to the boundary condition on the center conductor are equal.

Gauss' Law provides for the last step toward the solution of the transmission line parameters. Keeping in mind such relations as

$$\varepsilon\oint \vec{E}\bullet d\vec{S} = q = C\phi_c,$$

$$\vec{E} = -\nabla\phi,\ \ Z_0 = \sqrt{\frac{L}{C}}\ \text{ and }\ \nu = \frac{1}{LC},$$

the relationship between the electrical and thermal solutions becomes

$$\oint_C (\text{grad}\,\phi, \vec{n})dl = -\frac{C_1\phi_c}{\varepsilon_0\varepsilon_1} \tag{5}$$

and the thermal equivalent of Gauss' Law becomes

$$\lambda_d\oint_C (\text{grad}\,T, \vec{n})dl = -r_1\overline{I^2} \tag{6}$$

where

λ_d = thermal conductivity
r_1 = resistance of the center conductor
I = current carried by the center conductor

$$C_1 = \frac{1}{Z_0\nu} = \frac{\sqrt{\varepsilon_0\varepsilon_1\mu_0\mu_1}}{Z_0}$$

is the capacitance per unit length of the transmission line.[1] Because of Equation 4,

$$\oint_C \frac{(\text{grad}\,\phi, \vec{n})dl}{\phi_c} = -\frac{C_1}{\varepsilon_0\varepsilon_1} = -\frac{r_1}{\lambda_d T_{o(c)}}\overline{I^2} \tag{7}$$

and

$$T_{o(c)} = \frac{\varepsilon_0\varepsilon_1 r_1}{C_1\lambda_d}\overline{I^2} = \frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d}P_c \tag{8}$$

The result of Equation 8 relates temperature increase $T_{o(c)}$ to the transmission line properties such as Z_0 and $Z = \sqrt{\mu_0/\varepsilon_0}$ according to Equation 1.

THEORY — ADDING DIELECTRIC LOSS

Heat dissipation within S due to the dielectric loss tangent makes Equation 3 nonhomogeneous. The similarity between ϕ and T, which was relied on in the previous section, breaks down unless homogeneity can be restored. With the inclusion of heat generated in the dielectric region, Poisson's equation is

$$\Delta T = -\frac{q(S)}{\lambda_d} \tag{9}$$

where

$q(S) = \dfrac{\omega\varepsilon_0\varepsilon_2|E|^2}{2}$ (the energy density of dielectric heat loss for one unit length of transmission line in the area S)
λ_d = thermal conductivity of the dielectric
ω = angular frequency
ε_0 = dielectric permittivity of free space
ε_2 = imaginary part of the dielectric constant (equivalent of loss tangent)
E = E(S) (the electric field in the area S)

E follows the definition

$$E = -\text{grad}\,\phi \tag{10}$$

The thickness δ_{ic} and thermal conductivity λ_{ic} of the inner conductor are assumed to be low and high, respectively, so that the temperature across the inner conductor is constant and equal to the temperature increase T_o. It is also assumed that the cooling of the external conductor and δ_{ec} and λ_{ec} are such that the region of the external conductor is at a constant temperature as well. (It is evidently the case that only unusual values of δ_{ec} and λ_{ec} could not warrant the constancy of temperature on the entire area of the external conductor. The microstripline with one-way cooling[4] could be an example.) When the region of the external conductor is at a constant temperature, the boundary conditions can be expressed as

$$T\big|_{ib} = T_o \text{ and } T\big|_{eb} = 0 \qquad (11)$$

Following Yurov[3] and starting with what is generally known as Green's first identity,

$$\text{div}(\phi \text{ grad } \phi) = \phi\Delta\phi + \left|\text{grad } \phi\right|^2 \qquad (12)$$

as well as taking into account Equations 2, 10 and 12, the expression for q(S) becomes

$$\begin{aligned} q(S) &= \frac{\omega\varepsilon_0\varepsilon_2\left|\text{grad}\,\phi\right|^2}{2} \\ &= \frac{\omega\varepsilon_0\varepsilon_2}{2}\text{div}(\phi\,\text{grad}\,\phi) \\ &= \frac{\omega\varepsilon_0\varepsilon_2}{2}\text{div}\left(\frac{\text{grad}\,\phi^2}{2}\right) \\ &= \frac{\omega\varepsilon_0\varepsilon_2}{4}\Delta\phi^2 \\ &= \Delta\left(\frac{1}{4}\omega\varepsilon_0\varepsilon_2\phi^2\right) \end{aligned} \qquad (13)$$

Equation 9 now can be rewritten in homogeneous form as

$$\Delta\Phi = 0 \text{ with } \Phi\big|_{ic} = T_o \text{ and } \Phi\big|_{ec} = 0 \qquad (14)$$

where

$$\Phi = T + \frac{\omega\varepsilon_0\varepsilon_2}{4\lambda_d}\phi^2$$

(the electrothermal potential)[3]

The solution Φ is the superposition of the solution for conductor loss only, T, and the solution for dielectric loss only. The two solutions each have an associated boundary condition, the sum of which provides the overall boundary condition $T_o = T_{o(c)} + T_{o(d)}$, which now can be written as

$$T_o = \frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d}P_c + \frac{\omega\varepsilon_0\varepsilon_2\phi_{ic}^2}{4\lambda_d} \qquad (15)$$

To solve for the power dissipation in the dielectric (analogous to P_c for the inner conductor), q(S) is integrated in area S. Use is made of the two-dimensional form of the divergence theorem and, by introducing a cut, the two adjacent sides on which the closed integral cancels. In addition, $\phi_{ec} = 0$.

$$\begin{aligned} P_d &= \int q(S)dS \\ &= \frac{\omega\varepsilon_2\varepsilon_0}{2}\int \text{div}(\phi\,\text{grad}\,\phi)dS \\ &= \oint_I (\phi\,\text{grad}\,\phi, \vec{n})dl \\ &+ \oint_O (\phi\,\text{grad}\,\phi, \vec{n})dl \\ &= \phi_{ic}\oint_I (\text{grad}\,\phi, \vec{n})dl \\ &+ \phi_{ec}\oint_O (\text{grad}\,\phi, \vec{n})dl \\ &= \frac{\omega\varepsilon_2\varepsilon_0}{2}\phi_{ic}(\text{grad}\,\phi, \vec{n})dl \end{aligned}$$

Taking Equation 5 into consideration and noting that the outward normal on C is opposite to that on I,

$$P_d = \frac{\omega\varepsilon_2\varepsilon_0 C_1}{2\varepsilon_1\varepsilon_0}\phi_{ic}^2 \qquad (16)$$

Inserting the expression for C_1, the main result becomes

$$T_o = \frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d}P_c + \frac{1}{2}\frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d}P_d \qquad (17)$$

For most transmission lines $\mu_1 = 1$, and so

$$\begin{aligned} Z &= \sqrt{\frac{\mu_0}{\varepsilon_0}} \\ &= 120\pi\,\Omega \end{aligned} \qquad (18)$$

Comparing Equations 17 and 1,

$$R_{o(c)} = MZ_0 \qquad (19)$$

$$R_{o(d)} = \frac{1}{2}R_{o(c)} \qquad (20)$$

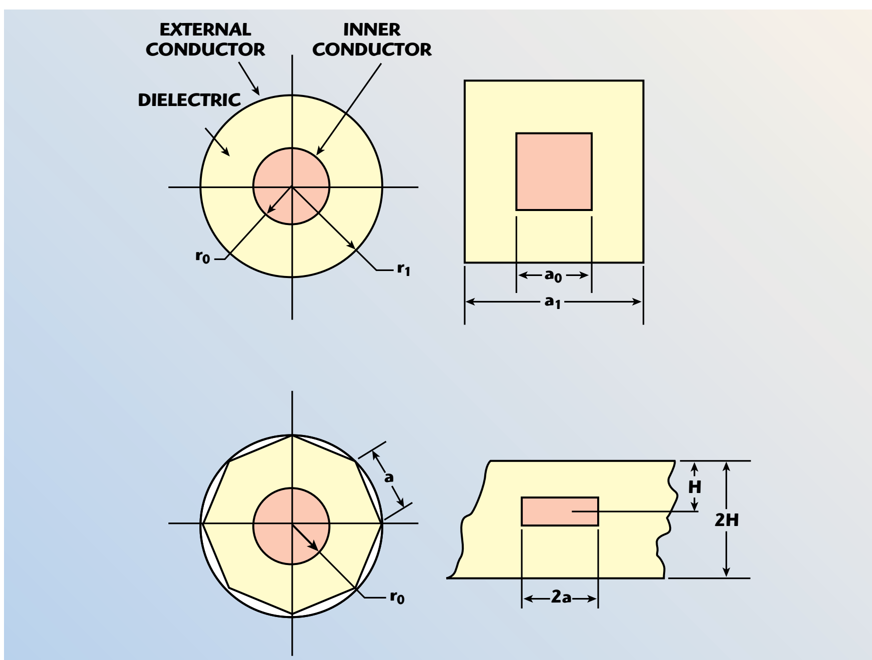
where

$$M = \frac{\sqrt{\varepsilon_1}}{120\pi\lambda_d}$$

(the scale factor depending on the physical parameters of dielectric, but not on the form and size of the feeder line)

It should be stressed that the simple and exact correlations achieved between Equations 19 and 20 are correct for every transmission line only when the inner and external conductors are the equal potential and isothermal surfaces, and the potential and temperature boundary conditions are alike.

Figure 1 shows some cases of using the correlation found in Equation



▲ *Fig. 1 The crosscut of some transmission lines.*

19 that will be reviewed. The first case is a coaxial line with conductors of round cross section with radii r_1 and r_0 ($r_1 > r_0$). An exact expression for this case is[1]

$$Z_0\sqrt{\varepsilon_1} = \frac{Z}{2\pi}\ln\frac{r_1}{r_0}$$

Thus, in compliance with Equation 19,

$$R_{o(c)} = \frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d} = \frac{1}{2\pi\lambda_d}\ln\frac{r_1}{r_0}$$

which is in agreement with the famous expression for the thermal impedance.[2]

The coaxial line with the conductors of square section having the sizes of the arm of the square $a_1 > a_0$ is the next case. The exact expression for $Z_0\sqrt{\varepsilon_1}$ obtained using the conformal transformation is[1]

$$Z_0\sqrt{\varepsilon_1} = \frac{1}{4}\frac{K(k)}{K'(k)}$$

where

K(k) = full elliptic integral of the first type

The value K/K' is identified in the literature.[1] For the situation in question, Wong[2] recommends the expression

$$R_{o(c)} = \frac{0.9252}{2\pi\lambda_d}\left(\ln\frac{a_1}{a_0} - 0.054\right), \quad \frac{a_1}{a_0} > 1.7$$

The departure of the calculation results produced by the formula for $R_{o(c)}$ from the exact result is $\delta = -3.3$ to -6.6%. When $a_1/a_0 \to 1$, this formula becomes meaningless.

The main round conductor of the r_0 radius coaxial with the external conductor, made as a regular polygon (n-number of arms) with each arm equal to a, is the third case. For this situation an approximate formula that exists in the literature[2] is

$$R_{o(c)} = \frac{1}{2\pi\lambda_d}\ln\left[\left(0.18n - 0.19\right)\frac{a}{r_0}\right]$$

(The accuracy is not taken into account.) Multiplying the above equation by $Z\lambda_d$ yields the expression for the $Z_0\sqrt{\varepsilon_1}$.

$$Z_0\left(\frac{a}{r_0}\right)\sqrt{\varepsilon_1}$$

exists in the literature[1] only for n = 3, 4, 5 and 6 and there is no equation for design.

The symmetrical stripline with the strip of zero thickness is the final case. The distance between the ground planes is 2H and the width of the strip is 2a. In the literature[1] there is an exact expression for $Z_0\sqrt{\varepsilon_1}$, which can be transformed using the equation for $Z\lambda_d$ after dividing by $R_{o(c)}$ such that

$$R_{o(c)} = \frac{1}{4\lambda_d}\frac{K(k)}{K'(k)}$$

In addition, approximation formulas exist that deviate from the exact solution by no more than 0.5 percent. For this case and a great number of others that do not fit the examples of the crosscut for which there are exact data,[1] there is no such information available in the literature.[2] Therefore, the way it was shown in the examples (using the correlation of Equation 19 and equations or tabs that are well known in microwave engineering for calculating Z_0) helps to provide at least one of the following results: to determine the unknown correlation for calculating $R_{o(c)}$, to expand the scope of parameters in which the value $R_{o(c)}$ can be calculated, to provide an approximate gauge for estimating the accuracy of the approximation models for calculating $R_{o(c)}$ and to establish the expression for $R_{o(c)}$ in a new form.

On the other hand, the correlation of Equation 19 could be used for establishing a new expression or elaborating on the approximation expression for Z_0 based on the solutions for $R_{o(c)}$ (see example 3). It should be stressed that in case the exact calculations for Z_0 and $R_{o(c)}$ agree precisely with the coefficient $Z\lambda_d/\sqrt{\varepsilon_1}$, the approximate solution for Z_0 and $R_{o(c)}$ in the same systems in microwave engineering and thermophysics traditionally can exist in different forms and methodologies. Thus, for cases 2 and 4 the approximate expressions for Z_0 have been obtained from the exact calculations using the approximation of the elliptic functions in Gunston;[1] however, in Shiffer[4] it was obtained using the exact solution of the equation of the thermal conductivity for the thermal equivalent's approximate model.

The thermal equivalent comprises simple physics principles and the equation for it is easily determined in comparison with virgin systems of bodies, especially when there are multiple systems having a simple form and the error of the solution happens to be acceptable for practice (and can be estimated beforehand). In this way, using the correlation of Equation 19 when deciding the problems connected with microwave engineering and thermophysics can enrich these fields.

Consider one more method of expressing T_o that is comfortable for practical calculations. From Equation 17,

$$T_o = \frac{Z_0\sqrt{\varepsilon_1}}{Z\lambda_d}\left(P_c + \frac{1}{2}P_d\right) = Z_0M\left(P_c + \frac{1}{2}P_d\right) \qquad (21)$$

or

$$T_o = Z_0M(2\alpha_{ic} + \alpha_d)P \qquad (22)$$

α_{ic} and α_d are the decrements caused by the losses in the inner conductor and volumetric losses in the dielectric area (Neper/m), and P is the average power in the cut area of the strip in question. Decrements α_{ic} and α_d were used in Equation 22 referring to their definition:

$$\alpha_{ic} = \frac{P_c}{2P}, \quad \alpha_d = \frac{P_d}{2P}$$

Using Equation 22, T_o is calculated for the line with unknown factor α_d by taking into account theoretical and experimental data for total loss α_Σ such that

$$\alpha_\Sigma = \alpha_c + \alpha_d \qquad (23)$$

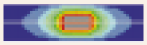
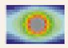
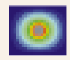
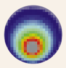
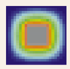
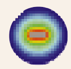
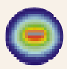
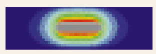
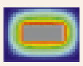
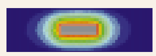
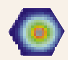
To determine T_o, Equation 22 is transformed keeping in mind that all transmission lines with the basic TEM-mode yield

$$\alpha_d = \frac{\omega\varepsilon_0\varepsilon_2}{2}Z = \frac{\pi\sqrt{\varepsilon_1}\,t_g\delta}{\lambda} \qquad (24)$$

λ is the length of the wave and $t_g\delta$ is the dielectric loss tangent of the ma-

TABLE I

POWER HANDLING CAPABILITIES OF TRANSMISSION LINES

Cross Section	Type	Cross Section	Type	Cross Section	Type
	High Q Triplate		Unshielded Slab Line		Shielded Slab Line
	Eccentric Coaxial Line		Square Coaxial Line		Coaxial Strip Line
	Coaxial Stripline		Triplate Line with Rounded Off Strip		
	Rectangular Coaxial Line		Triplate Stripline		Hexagon Coaxial Line

terial, filling the line

$$T_o = Z_0 I\left(2\alpha_\Sigma - \frac{\omega\varepsilon_0\varepsilon_2 Z}{2}\right)P$$

or

$$T_o = Z_0 M\left(2\alpha_\Sigma - \frac{\pi\sqrt{\varepsilon_1}\,t_g\delta}{\lambda}\right)P \quad (25)$$

Two more cases will be examined using the results received.

Case 5 involves the central circular conductor of the r_0 radius coaxial with the external conductor that is made in the configuration of the regular heptagon with the arm a. The approximate formula for $R_{o(c)}$ is written as[2]

$$R_{o(c)} = \frac{Z}{2\pi\lambda_d}\ln\left(1.07\frac{a}{r_0}\right)$$

Making use of the correlations of Equations 19 and 20 yields

$$Z_0 = \frac{Z}{2\pi\sqrt{\varepsilon_1}}\ln\left(1.07\frac{a}{r_0}\right)$$

$$R_{o(d)} = \frac{1}{4\pi\lambda_d}\ln\left(1.07\frac{a}{r_0}\right)$$

Thus, the value of maximum overheat with the help of Equation 21 yields

$$T_o = \frac{1}{2\pi\lambda_d}\left[\ln\left(1.07\frac{a}{r_0}\right)\right]\left(P_c + \frac{1}{2}P_d\right)$$

When the decrement of the α_Σ line (obtained theoretically or experimentally) is known, the total allowable power P_{max} that the strip can hold out is defined if the top allowable overhead is equal to T_o^{max} (for example, from the thermal condition of dielectric):

$$P_{max} = \frac{2\pi\lambda_d T_o^{max}}{2\alpha_\Sigma - \frac{\pi\sqrt{\varepsilon_1}\,t_g\delta}{\lambda}\ln\left(1.07\frac{a}{r_0}\right)}$$

Case 6 involves a symmetrical stripline with the symmetrical cooling of the ground planes. The exact values of Z_0 and α_c are known for this line as

$$Z_0 = \frac{30\pi K(k)}{\sqrt{\varepsilon_1}K(k')}$$

$$\alpha_{ic} = \alpha_s\frac{\left(C_p + 2C_f\right)}{2\left(C_p + C_f\right)}$$

where

$K(k)$ = full elliptic integral of the first type

The connection k, k', C_p and C_f with the sizes of the line and the expression for α_s are provided by Shiffer.[4] Therefore, the exact expression for T_o using Z_0 and α_c is

$$T_o = \frac{K(k)}{4\lambda_d K(k')}\left(2\alpha_{ic} + \frac{\pi\sqrt{\varepsilon_1}\,t_g\delta}{\lambda}\right)P \quad (26)$$

The correlation of Equation 26 allows the exact calculation of the maximum overheat of the symmetrical stripline.

It should be noted that Shiffer[4] provides only approximate estimates of T_o, which are usually decreased by one and a half to two times because of the approximation of the calculation. ***Table 1*** lists examples of the power handling capabilities of various transmission lines.

CONCLUSION

The connection between thermal resistance and impedance has been reviewed. The scale factors depend only on physical qualities of the dielectric material. With the help of the scale factors using known solutions of the electrodynamical task, all impedances that are necessary for calculation of thermal behavior can be determined. Moreover, if the thermal resistance is known, it is possible to calculate the wave impedance of the microwave transmission line. The exact correlation that is obtained allows the calculation of maximum overheat of the inner conductor T_o compared to the field in the known case. In addition, the wave impedance Z_0 and the losses in the inner conductor P_{ic} and dielectric P_d can be determined along with the decrements α_c in the conductor and α_d in the dielectric and the theoretical or experimental value of the full decrement α_Σ. ■

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